

Partial Solution Set, Leon §6.4

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6.4.1a For  $\mathbf{z} = \begin{bmatrix} 4+2i \\ 4i \end{bmatrix}$  and  $\mathbf{w} = \begin{bmatrix} -2 \\ 2+i \end{bmatrix}$ , compute  $\|\mathbf{z}\|$ ,  $\|\mathbf{w}\|$ ,  $\langle \mathbf{z}, \mathbf{w} \rangle$ , and  $\langle \mathbf{w}, \mathbf{z} \rangle$ .

**Solution:**  $\|\mathbf{z}\| = \sqrt{\mathbf{z}^H \mathbf{z}} = \sqrt{36} = 6$ ,  $\|\mathbf{w}\| = \sqrt{\mathbf{w}^H \mathbf{w}} = \sqrt{9} = 3$ ,  $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H \mathbf{z} = -4 + 4i$ , and  $\langle \mathbf{w}, \mathbf{z} \rangle = \mathbf{z}^H \mathbf{w} = -4 - 4i$ .

6.4.2b Let  $\mathbf{z}_1 = \begin{bmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{bmatrix}$ , and  $\mathbf{z}_2 = \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$ . Write the vector  $\mathbf{z} = \begin{bmatrix} 2+4i \\ -2i \end{bmatrix}$  as a linear combination of  $\mathbf{z}_1$  and  $\mathbf{z}_2$ .

**Solution:** From part (a) of this exercise, we know that  $\{\mathbf{z}_1, \mathbf{z}_2\}$  is an orthonormal set, so we don't have to work very hard to come up with coefficients  $c_1, c_2$  such that  $\mathbf{z} = c_1 \mathbf{z}_1 + c_2 \mathbf{z}_2$ . By Theorem 5.5.2 and the definition of the complex inner product,  $c_1 = \langle \mathbf{z}, \mathbf{z}_1 \rangle = 4$ , and  $c_2 = \langle \mathbf{z}, \mathbf{z}_2 \rangle = 2\sqrt{2}$ .

6.4.3 Let  $\{\mathbf{u}_1, \mathbf{u}_2\}$  be an orthonormal basis for  $\mathbf{C}^2$ , and let  $\mathbf{z} = (4+2i)\mathbf{u}_1 + (6-5i)\mathbf{u}_2$ .

(a) What are the values of  $\mathbf{u}_1^H \mathbf{z}$ ,  $\mathbf{z}^H \mathbf{u}_1$ ,  $\mathbf{u}_2^H \mathbf{z}$ , and  $\mathbf{z}^H \mathbf{u}_2$ ?

**Solution:**

$$\begin{aligned} \mathbf{u}_1^H \mathbf{z} &= \mathbf{u}_1^H ((4+2i)\mathbf{u}_1 + (6-5i)\mathbf{u}_2) \\ &= (4+2i)\mathbf{u}_1^H \mathbf{u}_1 + (6-5i)\mathbf{u}_1^H \mathbf{u}_2 \\ &= 4+2i. \end{aligned}$$

Similarly  $\mathbf{z}^H \mathbf{u}_1 = \overline{\mathbf{u}_1^H \mathbf{z}} = 4-2i$ ,  $\mathbf{u}_2^H \mathbf{z} = 6-5i$ , and  $\mathbf{z}^H \mathbf{u}_2 = \overline{\mathbf{u}_2^H \mathbf{z}} = 6+5i$ .

(b) What is the value of  $\|\mathbf{z}\|$ ?

**Solution:**  $\|\mathbf{z}\|^2 = \mathbf{z}^H \mathbf{z} = (4+2i)(4-2i) + (6-5i)(6+5i) = 16+4+36+25 = 81$ , so  $\|\mathbf{z}\| = 9$ .

6.4.5 Find an orthogonal or unitary diagonalizing matrix for each of the following matrices.

(a)  $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ .

**Solution:**  $A$  is real and symmetric, so we know that if  $A$  has distinct eigenvalues then the corresponding eigenvectors are orthogonal. The eigenvalues turn out to be  $\lambda_1 = 1$  and  $\lambda_2 = 3$ . The corresponding eigenvectors may be taken to be  $\mathbf{u}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{-1} \end{bmatrix}$  and  $\mathbf{u}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{\sqrt{2}}{1} \end{bmatrix}$ , and a diagonalizing matrix for  $A$  is  $U = [\mathbf{u}_1 \quad \mathbf{u}_2]$ .

(c)  $A = \begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ . The eigenvalues of  $A$  are  $\lambda_1 = 1$ ,  $\lambda_2 = 2$ , and  $\lambda_3 = 3$ . The corresponding unit eigenvectors may be taken to be

$$\mathbf{u}_1 = \begin{bmatrix} \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \mathbf{u}_3 = \begin{bmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix},$$

and the diagonalizing matrix is  $U = [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]$ . It's a good idea to check:

$$\begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{-i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

6.4.10 Given  $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}$ , find a matrix  $B$  such that  $B^H B = A$ .

**Solution:** Since  $A$  is Hermitian, it follows that  $A$  is diagonalizable by a unitary matrix  $U$ , i.e.,  $A = UDU^H$ . Letting  $B = D^{1/2}U^H$ , we have

$$B^H B = U \overline{D^{1/2}} D^{1/2} U^H = UDU^H = A.$$

In this particular case, we may take  $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i/\sqrt{2} & -i/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$  and  $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,

from which we obtain  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -i & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .

6.4.11 Let  $U$  be a unitary matrix. Prove:

- (a)  $U$  is normal.
- (b)  $\|U\mathbf{x}\| = \|\mathbf{x}\|$  for all  $\mathbf{x} \in \mathbf{C}^n$ .
- (c) If  $\lambda$  is an eigenvalue of  $U$ , then  $|\lambda| = 1$ .

**Solution:**

- (a) Since  $U^{-1} = U^H$ , then  $U^H U = U U^H$ , and so  $U$  is normal.

(b) It suffices to show that  $\|U\mathbf{x}\|^2 = \|\mathbf{x}\|^2$  for all  $\mathbf{x} \in \mathbf{C}^n$ . So let  $\mathbf{x} \in \mathbf{C}^n$ . Then

$$\|U\mathbf{x}\|^2 = (U\mathbf{x})^H(U\mathbf{x}) = \mathbf{x}^H U^H U \mathbf{x} = \mathbf{x}^H \mathbf{x} = \|\mathbf{x}\|^2$$

(c) Let  $\mathbf{x}$  be an eigenvector for  $U$ , with associated eigenvalue  $\lambda$ . Then  $U\mathbf{x} = \lambda\mathbf{x}$ , so  $\|U\mathbf{x}\| = \|\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ , and it follows that  $|\lambda| = 1$ .